Mutual Coupling Between Parallel-Plate Waveguides

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Abstract—The radiation field and mutual coupling between two identical parallel-plate waveguides having the same axis of symmetry are investigated. Jones' method of formulation is applied and a modified Wiener–Hopf equation is obtained. Expressions for the radiated field in free space, reflected field in the exciting waveguide, and transmitted field in the coupled waveguide are obtained and the reflected and transmitted fields are expressed in terms of waveguide modes. The reflection coefficient for each mode is represented by three terms, two of which are due to reflections at the open end of the exciting waveguide and are constant along the waveguide. The third term is the contribution from the field scattered by the open end of the coupled waveguide and decays along the waveguide according to the radiation condition. Similarly, the transmission coefficient of each mode is represented by three terms, two of which decay along the coupled waveguide and the third one is constant. The radiation field is also divided into three terms. One of them is due to the radiation from the open end of the exciting waveguide and the other two are the contribution of multiple interactions between the two waveguides.

Computed results for the reflection and transmission coefficients and the radiation field are shown for $TE_{0,1}$ excitation and various separation distance of the waveguides. The results for the reflection and transmission coefficients are oscillating functions of period π, and approach gradually the well-known final values of a single excited waveguide.

I. Introduction

RECENTLY, open-ended waveguide structures have received considerable attention due to their importance as radiating elements [1]–[3] or microwave measurement devices [4]. The previous analytical investigation of these structures is mostly based on the equivalent static approach [5] and the ray theory of diffraction [6]. The equivalent static approach has been used to study various waveguide geometries. But its applicability is limited to the wavelength range, where the higher order diffraction fields can not propagate. Similarly, the ray theory of diffraction has been used to study similar problems, in particular the mutual coupling between parallel-plate waveguides [7] and horn antennas [8]. Its application is also limited to certain waveguide geometries due to difficulties in including whole rays.

For problems concerning symmetrical geometries, an alternative method based on the Wiener–Hopf technique is usually used to solve the resulting symmetrical boundary value problems. This paper considers the boundary value problems concerning two parallel-plate waveguides, having the same width and axis of symmetry. Thus field equations are utilized to derive a modified Wiener–Hopf equation, similar to that of [9]–[12]. The final results are expressed in terms of an integral extending from zero to infinity, but suitable for numerical integration [13] using a Gauss–Laguerre quadrature formula [14].

In order to reduce the solution to that of ray theory of diffraction, the integral in the final expressions is approximated by expanding the transformed Green's function $G(\alpha)$ in a power series and retaining the first term only. Consequently, the results after integration are in terms of a series convergent for $[(ka)^2/kL]<1$, where a and L are the width and separation distance of waveguides and k is the propagation constant of free space. This is the same condition which Kashyab and Hamid [12] have used in investigating the diffraction characteristics of a similar geometry. The final solutions, both rigorous and that obtained by the ray theory of diffraction in conjunction with the modified diffraction coefficient of Lee [15] and [16], are divided into three terms. The first term represents the solution due to the exciting waveguide alone, while the second and third terms are the contribution of multiple diffractions between the exciting and coupled waveguides.

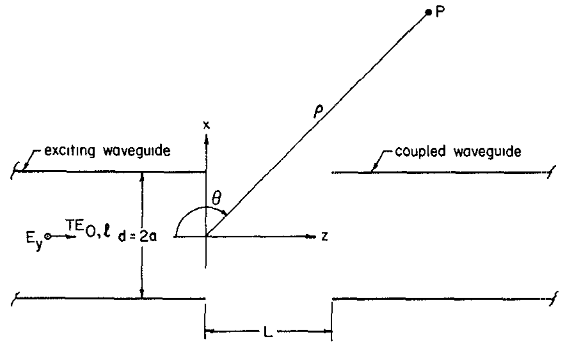

Fig. 1. Geometry of the problem.

II. Formulation of the Problem

Consider two infinitely thin and perfectly conducting parallel-plate waveguides, having width $2a$ and separated by a distance L, located in free space as shown in Fig. 1. With a time factor $e^{-i\omega t}$, an incident field consisting of a $TE_{0,l}$ mode is assumed to be propagating in the (exciting) waveguide along the positive z direction, in the form

$$E_y{}^i = \phi^i(x,z) = \cos\left(\frac{l\pi x}{2a}\right)e^{-\gamma_l z}, \qquad l = 1,3,5\cdots \tag{1}$$

where $\gamma_l=[(l\pi/2a)^2-k^2]^{1/2}$ and $k=k_1+ik_2$ is the propagation consistant in free space. The resulting total EM fields may be found from $\phi^t=\phi^i+\phi$, where ϕ is the scattered field and satisfies a two-dimensional wave equation and should be solved subject to the appropriate boundary and edge conditions [17]. Using Jones' method of formulation [9], the following modified Wiener–Hopf equation of second type [18], [19] is obtained.

$$J_-(\alpha) + e^{i\alpha L}J_+(\alpha) + \Phi_1(a,\alpha)/G(\alpha)$$
$$= \frac{i\pi l}{2a\sqrt{2\pi}}(-1)^{(l-1)/2}\frac{1-e^{(i\alpha-\gamma_l)L}}{\alpha+i\gamma_l}, \qquad |\tau| < k_2. \tag{2}$$

The unknown $\Phi_1(a,\alpha)$ is a finite-range transform

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$$\Phi_1(a, \alpha) = \frac{1}{\sqrt{2\pi}} \int_0^L \phi(x, z) e^{i\alpha z} \, dz \tag{3}$$

where $\alpha = \sigma + i\tau$ is the Fourier transform variable, and $G(\alpha) = \cosh \gamma a / \gamma a \exp(\gamma a)$ is the transformed Green's function associated with the Wiener–Hopf equation. $J_+(\alpha)$ and $J_-(\alpha)$ are unknown and are analytic in the upper $(\tau > -k_2)$ and lower $(\tau < k_2)$ halves of the α plane, respectively. It can be shown that $\Phi_1(a, \alpha)$ satisfies [19].

$$\Phi_1(a, \alpha) = \tfrac{1}{2} G(\alpha) \left[e^{i\alpha L} \{ S(-\alpha) - D(-\alpha) \} - \{ S(\alpha) + D(\alpha) \} \right] \tag{4}$$

where

$$\left. \begin{matrix} S(\alpha) \\ D(\alpha) \end{matrix} \right\} = J_-(\alpha) \mp J_+(-\alpha) - \frac{i\pi l}{2a\sqrt{2\pi}} (-1)^{(l-1)/2} \cdot \left[\frac{1}{\alpha + i\gamma_l} \mp \frac{e^{-\gamma_l L}}{\alpha - i\gamma_l} \right]. \tag{5}$$

These functions satisfy the following integral equation:

$$\frac{i\pi l}{2a\sqrt{2\pi}} (-1)^{(l-1)/2} \frac{G_+(i\gamma_l)}{\alpha + i\gamma_l} + G_-(\alpha) E(\alpha) = \frac{\lambda}{2\pi i} \int_{-\infty - id}^{\infty - id} \frac{G_+(\beta) E(\beta) e^{-i\beta L}}{\beta + \alpha} \, d\beta, \quad -k_2 < -d < \tau < d < k_2 \tag{6}$$

where

$$E(\alpha) = \begin{cases} S(\alpha), & \lambda = 1 \\ D(\alpha), & \lambda = -1 \end{cases} \tag{7}$$

and $G_+(\alpha)$ is the "plus part" of $G(\alpha)$ $(G(\alpha) = G_+(\alpha) G_-(\alpha))$ and is given by [19]

$$G_+(\alpha) = G_-(-\alpha) = \sqrt{\frac{\cos ka}{k + \alpha}} \, e^{i(\pi/4)} e^{i(\alpha a/\pi)[1 - C + \ln(2\pi/ka) + i(\pi/2)]} \cdot e^{i(\gamma a/\pi) \ln (\alpha - \gamma)/k} \prod_{n=1,3,5,\cdots}^{\infty} \left(1 + \frac{\alpha}{i\gamma_n}\right) e^{i(2\alpha a/n\pi)} \tag{8}$$

where $C = 0.57721 \cdots$ is Euler's constant, and

$$'\gamma_n = \left[\left(\frac{n\pi}{2a} \right)^2 - k^2 \right]^{1/2}.$$

A solution of the integral equation (6) together with (4) gives $\Phi_1(a, \alpha)$ and hence $\Phi(x, \alpha)$ can be determined. The final solution of $\phi(x, z)$ can be found by an inverse Fourier transform. To determine $E(\alpha)$ one notes that the right-hand side of (6) is of the form

$$I = \int_{-\infty - id}^{\infty - id} \frac{G_+(\beta) E(\beta)}{\alpha + \beta} e^{-i\beta L} \, d\beta = \int_{-\infty - id}^{\infty - id} \frac{a \cosh \gamma a \cdot e^{-\gamma a} E(\beta) e^{-i\beta L}}{\gamma a (\beta + \alpha) G_-(\beta)} \, d\beta \tag{9}$$

where $E(\alpha)$ denotes $S(\alpha)$ or $D(\alpha)$. For large L, the major contribution for the integral is from the integral over a small neighborhood around the branch point $\beta = -k$ [18]. The contour of integration may then be deformed in the lower half of the β plane, as shown in Fig. 2. An expansion of $G_-(\beta)$ and $E(\beta)$ in a Taylor series about the branch point $\beta = -k$ and retaining the first term only gives

$$I \simeq a \frac{E(-k)}{G_-(-k)} \int_p \frac{\cosh \gamma a}{\gamma a} e^{-\gamma a} \frac{e^{-i\beta L}}{\beta + \alpha} \, d\beta \tag{10}$$

where $p = p_1 + p_2 + p_3$. The integral over the small circle p_2 may be shown to be zero which reduces (9) to

$$I = a \frac{E(-k)}{G_-(-k)} T(\alpha) \tag{11}$$

where

$$T(\alpha) = 2 \int_{-k}^{-k - i\infty} \frac{\cosh^2 \gamma a}{\gamma a (\beta + \alpha)} e^{-i\beta L} \, d\beta. \tag{12}$$

Letting $\beta = -k - (iu/L)$ in the above equation gives

$$T(\alpha) = \frac{2L}{a} e^{ikL} \cdot \int_0^\infty \frac{\cosh^2 \left[\frac{a}{L} \sqrt{2ikL \cdot u - u^2} \right]}{\sqrt{2ikL \cdot u - u^2} \left[u + ikL \left(\frac{\alpha}{k} - 1 \right) \right]} e^{-u} \, du \tag{13}$$

which for a given value of α may be computed numerically using the Gauss–Laguerre quadrature formula. Substituting (11) into (6) one obtains

$$S(\alpha) = \frac{-i\pi l}{2a\sqrt{2\pi}} (-1)^{(l-1)/2} \frac{G_+(i\gamma_l)}{(\alpha + i\gamma_l) G_-(\alpha)} + \frac{a}{2\pi i} \frac{S(-k)}{G_+(k)} \frac{T(\alpha)}{G_-(\alpha)} \tag{14a}$$

$$D(\alpha) = \frac{-i\pi l}{2a\sqrt{2\pi}} (-1)^{(l-1)/2} \frac{G_+(i\gamma_l)}{(\alpha + i\gamma_l) G_-(\alpha)} - \frac{a}{2\pi i} \frac{D(-k)}{G_+(k)} \frac{T(\alpha)}{G_-(\alpha)}. \tag{14b}$$

Using these equations $\Phi_1(a, \alpha)$ in (4) can be expressed in terms of $S(-k)$ and $D(-k)$, which are known from (14a) and (14b) by letting $\alpha = -k$. Thus one finds

$$\Phi_1(a, \alpha) = \frac{i\pi l}{2a\sqrt{2\pi}} (-1)^{(l-1)/2} G_+(i\gamma_l) \cdot \left\{ \frac{G_+(\alpha)}{\alpha + i\gamma_l} - \frac{a/2\pi i}{(i\gamma_l - k)(1 - F^2) G_+^2(k)} \cdot \left[F T(\alpha) G_+(\alpha) + T(-\alpha) G_+(-\alpha) e^{i\alpha L} \right] \right\} \tag{15}$$

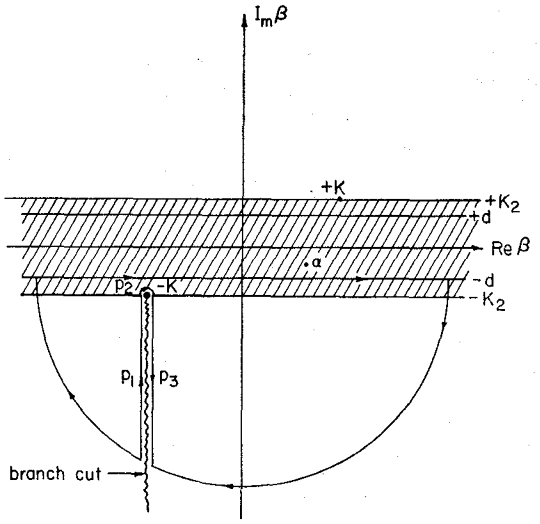

Fig. 2. Contour for the integral I in the β plane.

where

$$F = \frac{-a}{2\pi i} \frac{T(-k)}{G_+^2(k)} \cdot \tag{16}$$

This completes the solution of the modified Wiener–Hopf equation.

III. Evaluation of the Scattered Field

A. Radiation Field

In the region outside the waveguides, the scattered electric field is given by

$$\phi^s(x, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} \Phi_1(a, \alpha) e^{\gamma(a-x)-i\alpha z} \, d\alpha, \qquad |\tau| < k_2 \tag{17}$$

which by a saddle-point method of integration for the far-zone field ($k\rho \gg 1$) gives

$$\phi^s(\rho, \theta) = \frac{e^{i(k\rho-(\pi/4))}}{\sqrt{k\rho}} k \sin\theta \Phi_1(a, k\cos\theta) e^{-ika\sin\theta} \tag{18}$$

where ρ and θ are polar coordinates defined in Fig. 1. Replacing for $\Phi_1(a, k\cos\theta)$ from (15), the above equation becomes

$$\phi^s(\rho, \theta) = f(\rho, \theta) \left[\frac{G_+(k\cos\theta)}{k\cos\theta + i\gamma_l} - \left(\frac{a}{2\pi i}\right) \frac{FT(k\cos\theta)G_+(k\cos\theta)}{(1-F^2)(i\gamma_l - k)G_+^2(k)} - \left(\frac{a}{2\pi i}\right) \frac{T(-k\cos\theta)G_+(-k\cos\theta)}{(1-F^2)(i\gamma_l - k)G_+^2(k)} e^{ikL\cos\theta} \right] \tag{19}$$

where

$$f(\rho, \theta) = \frac{i\pi l}{2a\sqrt{2\pi k\rho}} (-1)^{(l-1)/2} G_+(i\gamma_l) k \cdot \sin\theta e^{i(k\rho - ka\sin\theta - \pi/4)}. \tag{20}$$

The radiation field in (19) consists of three terms. The first term is the well-known radiation field from the open end of a waveguide (in the absence of the coupled waveguide), whereas the second and third terms are the radiation fields due to the interactions between the two waveguides. More specifically, the second term gives the radiation from the open end of the exciting waveguide due to interaction with the coupled waveguide, with the third term being the radiation from the open end of the coupled waveguide due to interaction with the exciting waveguide.

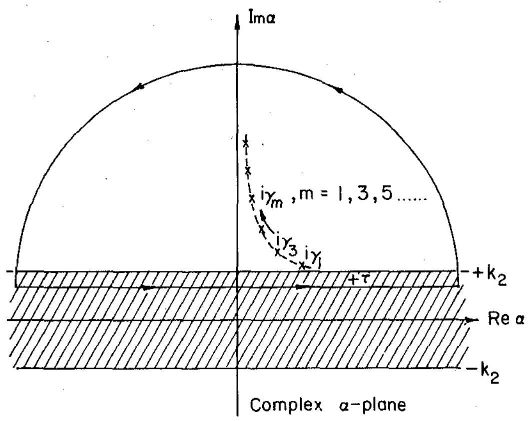

Fig. 3. Contour of integration for the first and second terms of $\Phi_1(a, \alpha)$ in (21).

B. Reflected Field

In the exciting waveguide ($z<0$), the reflected electric field is given by

$$\phi_r(x, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} \Phi_1(a, \alpha) \frac{\cosh\gamma x}{\cosh\gamma a} e^{-i\alpha z} \, d\alpha, \qquad |\tau| < k_2. \tag{21}$$

For the first two terms of $\Phi_1(a, \alpha)$, the contour of integration may be closed in the upper half of the complex α plane, as shown in Fig. 3. The only singularities so enclosed are the poles at $\alpha = i\gamma_m$, $\gamma_m = \sqrt{(m\pi/2a)^2 - k^2}$, where $m = 1, 3, 5, \cdots$. Thus one finds by the residue theorem, from the first term,

$$\phi_r^{\text{exc}}(x, z) = \sum_{m=1,3,5,\cdots}^{\infty} R_{l,m} \cos\left(\frac{m\pi}{2a} x\right) e^{\gamma_m z} \tag{22}$$

with

$$R_{l,m} = -\frac{l\pi^2}{4a^3} (-1)^{(l+m)/2} \frac{mG_+(i\gamma_l)G_+(i\gamma_m)}{\gamma_m(\gamma_m + \gamma_l)} \tag{23}$$

and from the second term,

$$\phi_{r,(1)}^{\text{int}}(x, z) = \sum_{m=1,3,5,\cdots}^{\infty} R_{l,m}^{(1)} \cos\left(\frac{m\pi}{2a} x\right) e^{\gamma_m z} \tag{24}$$

with

$$R_{l,m}^{(1)} = \frac{il\pi^2}{4a^3} (-1)^{(l+m)/2} \frac{mG_+(i\gamma_l)G_+(i\gamma_m)}{\gamma_m(i\gamma_l - k)G_+^2(k)} T(i\gamma_m) \cdot \left(\frac{a}{2\pi i}\right) \frac{F}{1-F^2} \cdot \tag{25}$$

The remaining term in $\Phi_1(a, \alpha)$ contributes

$$\phi_{r,(2)}{}^{\text{int}}(x, z) = \left(\frac{-il}{4a}\right)(-1)^{(l-1)/2}\frac{G_+(i\gamma_l)}{G_+^2(k)} \cdot \frac{(a/2\pi i)}{(i\gamma_l - k)(1-F^2)}\int_{-\infty+i\tau}^{\infty+i\tau} T(-\alpha)G_+(-\alpha) \cdot \frac{\cosh \gamma x}{\cosh \gamma a} e^{i\alpha(L-z)}\, d\alpha, \qquad |\tau| < k_2 \tag{26}$$

which may be expressed in a model series of the form

$$\phi_{r,(2)}{}^{\text{int}}(x, z) = \sum_{m=1,3,5,\cdots}^{\infty} R_{l,m}{}^{(2)}(z) \cos\left(\frac{m\pi}{2a}x\right). \tag{27}$$

Equating these equations, the orthogonality on x gives

$$R_{l,m}{}^{(2)}(z) = \frac{-i\pi lm}{4a^3}(-1)^{(l+m)/2}\frac{G_+(i\gamma_l)}{G_+^2(k)(i\gamma_l - k)(1-F^2)} \cdot \left(\frac{a}{2\pi i}\right)\int_{-\infty+i\tau}^{\infty+i\tau}\frac{T(-\alpha)\cosh\gamma a}{\gamma G_+(\alpha)(\alpha^2+\gamma_m{}^2)} e^{-\gamma a + i\alpha(L-z)}\, d\alpha. \tag{28}$$

In this equation, the contour of integration is closed in the upper half-plane. The only singularity so enclosed is the branch point at $\alpha = k$ which may be shown to give

$$R_{l,m}{}^{(2)}(z) = \frac{-i\pi lm}{2a^2}(-1)^{(l+m)/2}\frac{G_+(i\gamma_l)}{G_+^2(k)(i\gamma_l - k)(1-F^2)} \cdot \left(\frac{a}{2\pi i}\right)\int_{k}^{\infty+i\infty}\frac{T(-\alpha)\cosh^2 \gamma a e^{i\alpha(L-z)}}{\gamma a \cdot G_+(\alpha)\cdot(\alpha^2+\gamma_m{}^2)}\, d\alpha. \tag{29}$$

Since no analytical solution of this integral is known, it is modified by a change of variable to a more suitable form for numerical integration using the Gauss–Laguerre quadrature formula. The final form is

$$R_{l,m}{}^{(2)}(z) = \frac{\pi lm}{2a^3}(-1)^{(l+m)/2}\frac{G_+(i\gamma_l)e^{ik(L-z)}}{G_+^2(k)(i\gamma_l + k)(1-F^2)}\left(\frac{a}{2\pi i}\right) \cdot \int_0^\infty \frac{\cosh^2\left[\frac{a}{L-z}\sqrt{2ik(L-z)u - u^2}\right]}{\sqrt{2ik(L-z)u-u^2}\, G_+\left(k + \frac{iu}{L-z}\right)} \cdot \frac{T\left(-k\left[1+\frac{iu}{k(L-z)}\right]\right)}{k^2\left[1+\frac{iu}{k(L-z)}\right]^2 + \gamma_m{}^2} e^{-u}\, du. \tag{30}$$

Thus the reflected electric field is in the form

$$\phi_r(x, z) = \phi_r{}^{\text{exc}}(x, z) + \phi_{r,(1)}{}^{\text{int}}(x, z) + \phi_{r,(2)}{}^{\text{int}}(x, z) = \sum_{m=1,3,5,\cdots}^{\infty}\left[(R_{l,m} + R_{l,m}{}^{(1)})e^{\gamma_m z} + R_{l,m}{}^{(2)}(z)\right] \cdot \cos\left(\frac{m\pi}{2a}x\right) \tag{31}$$

where $R_{l,m}$, $R_{l,m}{}^{(1)}$, and $R_{l,m}{}^{(2)}(z)$ are the reflection coefficients given, respectively, by (23), (25), and (30).

Again, the reflected field is expressed by three terms. The first term gives the reflected field due to the open end of the exciting waveguide in the absence of the coupled waveguide. The remaining two terms are due to interactions. The second term is the contribution of the field scattered at the open end of the exciting waveguide when illuminated by the scattered field of the coupled waveguide, and the third term is due to the scattered field of the coupled waveguide, in the absence of exciting waveguide, when illuminated by the scattered field of the exciting waveguide. The reflection coefficients of this latter term thus represent a continuous spectrum of inhomogeneous plane waves which decay with z being zero at $z = -\infty$ according to the Sommerfeld radiation condition. It should be noted that for large values of L or z, the integral in (26) can be evaluated by the saddle-point method, to give the required contribution. The above method, however, is adopted to enable one to evaluate the resulting field for any given value of z, in particular the aperture field at $z=0$. Because of this attenuating nature of the last term, the reflected field at large distances from the opening is due to only $R_{l,m}$ and $R_{l,m}{}^{(1)}$.

C. Transmitted Field

In the coupled waveguide ($z > L$), the transmitted electric-field component is given by

$$\phi_t(x, z) = \phi^i(x, z) + \frac{1}{\sqrt{2\pi}}\int_{-\infty+i\tau}^{\infty+i\tau}\Phi_1(a, \alpha)\frac{\cosh \gamma x}{\cosh \gamma a} e^{-i\alpha z}\, d\alpha, \qquad |\tau| < k_2. \tag{32}$$

The integral may be evaluated by closing the contour in the lower half-plane. The first term of $\Phi_1(a, \alpha)$ has a pole at $\alpha = -i\gamma_l$ and a branch point at $\alpha = -k$. The contribution of the pole cancels the incident field exactly and the branch point contribution can be evaluated similar to $\phi_{r,(2)}{}^{\text{int}}(x, z)$. The result may be shown to be

$$\phi_t{}^{\text{ext}}(x, z) = \sum_{m=1,3,5,\cdots}^{\infty} T_{l,m}(z)\cos\left(\frac{m\pi}{2a}x\right) \tag{33}$$

where

$$T_{l,m}(z) = \frac{\pi lm}{2a^3}(-1)^{(l+m)/2} e^{ikz} G_+(i\gamma_l) \cdot \int_0^\infty \frac{\cosh^2\left[\frac{a}{z}\sqrt{2ikzu - u^2}\right]}{\sqrt{2ikzu - u^2}\, G_-\left(-k - \frac{iu}{z}\right)} \cdot \frac{e^{-u}}{\left[k^2\left(1+\frac{iu}{kz}\right)^2 + \gamma_m{}^2\right]\left[-k - \frac{iu}{z} + i\gamma_l\right]}\, du. \tag{34}$$

Similarly, for the second term of $\Phi_1(a, \alpha)$, the only enclosed singularity is the branch point $\alpha = -k$, and hence

$$\phi_{t,(1)}{}^{\text{int}}(x, z) = \sum_{m=1,3,5,\cdots}^{\infty} T_{l,m}{}^{(1)}(z)\cos\left(\frac{m\pi}{2a}x\right) \tag{35}$$

where

$$T_{l,m}^{(1)}(z) = \frac{-\pi lm}{2a^3}(-1)^{(l+m)/2}e^{ikz}\frac{FG_+(i\gamma_l)}{(1-F^2)(i\gamma_l-k)G_+^2(k)}\left(\frac{a}{2\pi i}\right) \cdot \int_0^\infty \frac{\cosh^2\left[\frac{a}{z}\sqrt{2ikzu-u^2}\right]\cdot T\left(-k-\frac{iu}{z}\right)}{\sqrt{2ikzu-u^2}\left[k^2\left(1+\frac{iu}{kz}\right)^2+\gamma_m^2\right]G_-\left(-k-\frac{iu}{z}\right)} e^{-u}\,du. \quad (36)$$

For the third term of $\Phi_1(a, \alpha)$, the enclosed singularities are the poles at $\alpha = -i\gamma_m$, with $\gamma_m = \sqrt{(m\pi/2a)^2 - k^2}$, where $m = 1, 3, 5, \cdots$. Evaluating these residue contributions, one obtains

$$\phi_{t,(2)}^{\text{int}} = \sum_{m=1,3,5,\cdots}^{\infty} T_{l,m}^{(2)} \cos\left(\frac{m\pi}{2a}x\right)e^{-\gamma_m z} \quad (37)$$

where

$$T_{l,m}^{(2)} = \frac{il\pi^2}{4a^3}(-1)^{(l+m)/2}\frac{mG_+(i\gamma_l)G_+(i\gamma_m)}{\gamma_m(1-F^2)(i\gamma_l-k)G_+^2(k)}T(i\gamma_m) \cdot \left(\frac{a}{2\pi i}\right)e^{\gamma_m L}. \quad (38)$$

Hence the transmitted electric field is given by

$$\phi_t(x, z) = \phi_t^{\text{ext}}(x, z) + \phi_{t,(1)}^{\text{int}}(x, z) + \phi_{t,(2)}^{\text{int}}(x, z) = \sum_{m=1,3,5}^{\infty}\left[T_{l,m}(z) + T_{l,m}^{(1)}(z) + T_{l,m}^{(2)}e^{-\gamma_m z}\right]\cos\left(\frac{m\pi}{2a}x\right) \quad (39)$$

where $T_{l,m}(z)$, $T_{l,m}^{(1)}(z)$, and $T_{l,m}^{(2)}$ are the transmission coefficients given, respectively, by (34), (36), and (38).

Again $T_{l,m}$ and $T_{l,m}^{(1)}(z)$ are expressed in convenient forms for numerical integration and may be computed using a Gauss–Laguerre quadrature formula to determine the aperture field. Furthermore they represent, respectively, the contribution of incident and scattered fields of coupled waveguide when scattered by the open end of the exciting waveguide. Thus they are decaying fields with z in accordance with the radiation condition. At large distances from the opening, the only transmission coefficient is $T_{l,m}^{(2)}$ which is due to the interaction between the two waveguides.

IV. Reduction of the Solution to that of Ray Theory of Diffraction

If the Green's function $G(\alpha)$ is expanded in a power series, then the function $T(\alpha)$ can be written as

$$T(\alpha) = \sum_{n=0,1,2,\cdots}^{\infty} T_n(\alpha) \quad (40)$$

where

$$T_n(\alpha) = \frac{1}{\epsilon_n}\int_{-\infty-id}^{\infty-id}\frac{(-2\gamma a)^n}{\gamma a\cdot n!}\,\frac{e^{-i\beta L}}{\beta+\alpha}\,d\beta \quad (41)$$

with

$$\epsilon_n = \begin{cases}1, & \text{for } n = 0\\ 2, & \text{for } n \neq 0.\end{cases}$$

In the neighborhood of $\beta = -k$, the function $(\beta - k)^{(n-1)/2}$ is regular and smooth, and can be replaced by $(-2k)^{(n-1)/2}$. Therefore (41), after deforming the contour, becomes

$$T_n(\alpha) = \begin{cases}\dfrac{(-1)^n 2^{n+1} a^{n-1}(2k)^{(n-1)/2}e^{-i(n-1)(\pi/2)}}{\epsilon_n\cdot n!}\displaystyle\int_{-k}^{-k-i\infty}(\beta+k)^{(n-1)/2}\frac{e^{-i\beta L}}{\beta+\alpha}\,d\beta, & n = 0, 2, 4, 6, \cdots \quad (42a)\\ 0 & n = 1, 3, 5, \cdots. \quad (42b)\end{cases}$$

A change of variable via $\beta = -k - (iu/L)$ gives the following:

$$T_n(\alpha) = \frac{(-1)^{n+1}2^{[(3n+1)/2]}(ka)^{n-1}e^{i(\pi/4)(n-1)}e^{ikL}}{\epsilon_n\cdot n!\cdot(kL)^{(n-1)/2}}W_{(n-2)/2}(\xi), \qquad n = 0, 2, 4, 6, \cdots \quad (43)$$

where

$$\xi = -iL(k-\alpha) \quad (44)$$

and

$$W_{j-(1/2)}(\xi) = \int_0^\infty \frac{u^j e^{-u}}{u+\xi}\,du. \quad (45)$$

The above function is related to the Whittaker function $W_{k,m}(\xi)$ by the relation

$$W_{j-(1/2)}(\xi) = \Gamma(j+1) \cdot \exp\left(\frac{\xi}{2}\right)\xi^{(j-1)/2}W_{-(1/2)(j+1),(j/2)}(\xi). \quad (46)$$

Using the asymptotic expansion of $W_{k,m}(\xi)$, [20], in (43), one obtains

$$T_n(\alpha) = \frac{(-1)^{n+1}2^{[(3n+1)/2]}(ka)^{n-1}e^{i(\pi/4)(n-1)}e^{ikL}}{\epsilon_n \cdot n! \cdot (kL)^{(n-1)/2}} \Gamma\left(\frac{n+1}{2}\right)\frac{1}{\xi} \cdot \left\{1 + \sum_{s=1}^{\infty} \frac{\left[\left(\frac{n-1}{4}\right)^2 - \left(\frac{n+3}{4}\right)^2\right]\left[\left(\frac{n-1}{4}\right)^2 - \left(\frac{n+7}{4}\right)^2\right]\cdots\left[\left(\frac{n-1}{4}\right)^2 - \left(\frac{n+4s-1}{4}\right)^2\right]}{s!\xi^s}\right\}. \quad (47)$$

Now retaining the first term in (47), its substitution into (40) gives

$$T(\alpha) = \sum_{n=0,2,4,\cdots}^{\infty} T_n(\alpha) = \frac{-i\sqrt{2}\, e^{i[kL-(\pi/4)]}}{a\sqrt{kL}\,(k-\alpha)} \cdot \sum_{n=0,2,4,\cdots}^{\infty} \frac{(-1)^n(2)^{3n/2}(ka)^n e^{i(n\pi/4)}}{\epsilon_n \cdot n!(kL)^{n/2}} \Gamma\left(n+\frac{1}{2}\right)$$
$$= \frac{-2\pi i e^{i[kL-(\pi/4)]}}{a\sqrt{2\pi kL}} \frac{1}{(k-\alpha)} \cdot \left[1 + i\nu - \nu^2 - \frac{2}{3}i\nu^3 + \cdots\right] \quad (48)$$

where $\nu = (ka)^2/kL$.

It is clear that for the convergence of $T(\alpha)$, ν must be less than unity, i.e., $(ka)^2 \ll kL$. Thus retaining the first term, (48) reduces to

$$T(\alpha) = T_0(\alpha) = \left(\frac{-2\pi i}{a}\right)\frac{e^{i[kL-(\pi/4)]}}{\sqrt{2\pi kL}}\frac{1}{k-\alpha}, \qquad L \text{ is large.} \quad (49)$$

Finally, this equation together with (16) gives

$$F = \frac{e^{i[kL-(\pi/4)]}}{2k\sqrt{2\pi kL}}\frac{1}{G_+^2(k)} \quad (50)$$

which is the same as $\overline{F}$ obtained in the next section.

A substitution of (49) and (50) into the expressions of the radiated, reflected, and transmitted fields gives the solutions which can be obtained using the ray theory of diffraction in conjunction with the modified diffraction coefficient [15], [16]. The details of the latter approach are shown in Section V. However, as $T_0(\alpha)$ yields the solution using ray theory of diffraction, the higher order terms of $T(\alpha)$ provide the correction when $(ka)^2/kL$ is not small enough.

V. Application of Ray Theory of Diffraction

S. W. Lee [15], [16] has introduced a modified diffraction coefficient for problems involving two or more parallel plates, which takes care of coupling along a shadow boundary. To apply the method to the present problem with an excitation of $TE_{0,l}$ mode and l odd, one utilizes the symmetry of the geometry with respect to z axis and introduces an infinitely large magnetic wall at the center of the waveguides, as shown in Fig. 4(a). The incident field is then a plane wave illuminating the upper edge of the exciting waveguide at an angle ϕ_l, where $\sin \phi_l = l\pi/2ka$. The resulting diffracted, reflected, and transmitted waves, then can be found by an application of the above modified diffraction coefficient.

A. Diffraction Patterns

Diffraction patterns consist of the diffraction due to the exciting waveguide alone and the multiple diffractions between plates 1 and 2, which may be considered separately as follows.

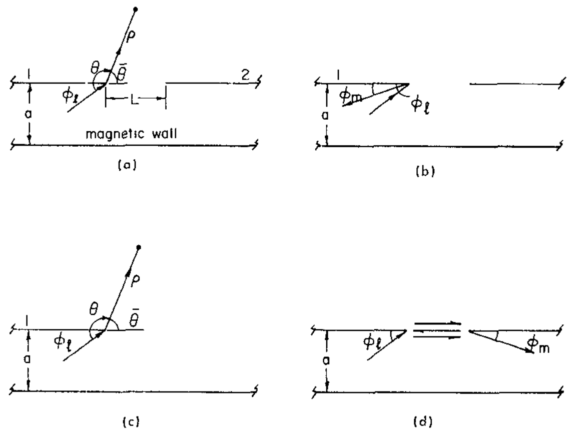

Fig. 4. Geometries for the application of the ray theory of diffraction.

1) Diffraction Due to the Open End of the Exciting Waveguide [Fig. 4(b)]: The field $\phi_1(\rho,\theta)$ on the ray diffracted at the edge of upper plate is given by

$$\phi_1(\rho,\theta) = \frac{e^{i[k\rho-(\pi/4)]}}{\sqrt{2\pi k\rho}}\overline{D}(\bar\theta,\theta_0)E_y^i, \qquad \bar\theta = \pi - \theta \quad (51)$$

where

$$E_y^i = \frac{i}{2}(-1)^{(l-1)/2} \quad (52)$$

at the upper edge, and θ_0 is the direction of the incident plane wave. ρ and θ are the coordinates of the observation point with respect to the upper edge and the factor $\overline{D}(\bar\theta,\theta_0)$ is the modified diffraction coefficient in the form

$$\overline{D}(\bar\theta,\theta_0) = \frac{-2i\cos\frac{\theta_0}{2}\cos\frac{\bar\theta}{2}}{\cos\theta_0 + \cos\bar\theta} f(\bar\theta)f(\theta_0) \quad (53)$$

with

$$f(\theta) = \begin{cases} \overline{G}_+(-k\cos\theta), & \frac{\pi}{2} < |\theta| < \pi \\ [\overline{G}_+(k\cos\theta)]^{-1}, & |\theta| < \frac{\pi}{2} \end{cases} \quad (54)$$

the function $\overline{G}_+$, used in [15] and [16], is related to G_+ of the present paper by

$$\overline{G}_+(\alpha) = \sqrt{2(\alpha+k)}\, e^{-i(\pi/4)}G_+(\alpha). \quad (55)$$

Thus for $\phi_1(\rho,\theta)$ one obtains

$$\phi_1(\rho,\theta) = \frac{e^{i[k\rho-(\pi/4)]}}{\sqrt{2\pi k\rho}}\left[\frac{i}{2}(-1)^{(l-1)/2}\right] \cdot \left[\frac{2k\sin\phi_l\sin\theta}{\cos\phi_l + \cos\theta}G_+(k\cos\phi_l)G_+(k\cos\theta)\right]. \quad (56)$$

Replacing $k \sin \phi_l$ by $l\pi/2a$ and $k \cos \phi_l = i\gamma_l$ in the above expression, it becomes

$$\phi_1(\rho, \theta) = \frac{i\pi l e^{i[k\rho-(\pi/4)]}}{2a\sqrt{2\pi k\rho}} (-1)^{(l-1)/2} G_+(i\gamma_l) \cdot k \sin\theta \frac{G_+(k\cos\theta)}{k\cos\theta + i\gamma_l}. \quad (57)$$

Note that, for $\theta < (\pi/2)$, the specular reflection at the magnetic wall requires the multiplication of the results by a factor of $[1+e^{2ika \sin \bar{\theta}}]$, which when combined by $[\bar{G}_+(k \cos \bar{\theta})]^{-1}$ gives $\bar{G}(-k \cos \theta)$. Thus for the range $0 < \bar{\theta} < \pi$ one can use a single expression $f(\bar{\theta}) = \bar{G}_+(-k \cos \bar{\theta})$ which gives the (57) for $\phi_1(\rho, \theta)$ valid for $0 < \theta < \pi$.

2) Multiple Diffraction Between Two Waveguides: There are two kinds of multiply-diffracted rays as shown in Table I with the integer n being at least unity. Now consider the fields due to rays of type

$$(A) = \sum_{n=1}^{\infty} \bar{F}_i \bar{F}^{(2n-1)} \bar{F}_{f_1} \quad (58)$$

where $\bar{F}_i$ is the diffraction field at edge 2 due to the initial diffraction of the incident plane wave at edge 1. $\bar{F}$ is the diffraction field at edge 1 or 2 with a plane wave of unit amplitude incident with angle zero at edge 2 or 1, respectively. $\bar{F}_{f_1}$ is the diffraction field, at the observation point, due to the final diffraction at edge 1 of an incident plane wave having a unit amplitude. Equation (58) can be written in the form

$$\phi_{2,A}(\rho, \theta) = \bar{F}_i \bar{F}_{f_1} \sum_{n=1}^{\infty} \bar{F}^{(2n-1)} = \bar{F}_i \bar{F}_{f_1} \frac{\bar{F}}{1-\bar{F}^2} \quad (59)$$

where $\bar{F}_i$, $\bar{F}$, and $\bar{F}_{f_1}$ are given by

$$\bar{F}_i = \frac{e^{i[kL-(\pi/4)]}}{\sqrt{2\pi kL}} \bar{D}(0, -(\pi - \phi_l)) E_y^i = \frac{-\pi l}{2a} (-1)^{(l-1)/2} \frac{e^{i[kL-(\pi/4)]}}{\sqrt{2\pi kL}} \frac{G_+(i\gamma_l)}{G_+(k)(i\gamma_l - k)} \quad (60)$$

$$\bar{F} = \frac{e^{i[kL-(\pi/4)]}}{\sqrt{2\pi kL}} \bar{D}(0,0) = \frac{e^{i[kL-(\pi/4)]}}{2k\sqrt{2\pi kL}} \frac{1}{G_+^2(k)} \quad (61)$$

$$\bar{F}_{f_1} = \frac{e^{i[k\rho-(\pi/4)]}}{\sqrt{2\pi k\rho}} \bar{D}(\bar{\theta}, 0) = \frac{-ie^{i[k\rho-(\pi/4)]}}{\sqrt{2\pi k\rho}} \frac{\sin\theta}{1-\cos\theta} \frac{G_+(k\cos\theta)}{G_+(k)}. \quad (62)$$

A substitution of (60) and (62) into (59) gives

$$\phi_{2,A}(\rho, \theta) = \frac{i\pi l(-1)^{(l-1)/2}}{2a\sqrt{2\pi k\rho}} e^{i[k\rho-(\pi/4)]} \frac{\bar{F}}{1-\bar{F}^2} \cdot \frac{G_+(i\gamma_l)\cdot k\sin\theta}{(i\gamma_l - k)G_+^2(k)} G_+(k\cos\theta) \cdot \left[\frac{e^{i[kL-(\pi/4)]}}{\sqrt{2\pi kL}} \frac{1}{k(1-\cos\theta)}\right]. \quad (63)$$

Similarly, fields due to rays of type B equal

$$\sum_{n=1}^{\infty} \bar{F}_i \bar{F}^{(2n-2)} \bar{F}_{f_2} e^{ikL\cos\theta}$$

with distances measured from edge 1 or

TABLE I

Type	Initial Diffraction at	Number of Diffraction at each edge		Final Diffraction at
		1	2	
A	1	n + 1	n	1
B	1	n	n	2

$$\phi_{2,B}(\rho, \theta) = \bar{F}_i \bar{F}_{f_2} \frac{1}{1-\bar{F}^2} e^{ikL\cos\theta} \quad (64)$$

where $\bar{F}_i$ and $\bar{F}$ are given by (60) and (61) and $\bar{F}_{f_2}$ is

$$\bar{F}_{f_2} = \frac{-ie^{i[k\rho-(\pi/4)]}}{\sqrt{2\pi k\rho}} \frac{\sin\theta}{1+\cos\theta} \frac{G_+(-k\cos\theta)}{G_+(k)}. \quad (65)$$

Again a substitution of (60) and (65) into (64) gives

$$\phi_{2,B}(\rho, \theta) = \frac{i\pi l(-1)^{(l-1)/2}}{2a\sqrt{2\pi kl}} e^{i[k\rho-(\pi/4)]} \frac{1}{1-\bar{F}^2} \cdot \frac{G_+(i\gamma_l)\cdot k\sin\theta}{(i\gamma_l - k)G_+^2(k)} G_+(-k\cos\theta) \cdot \left[\frac{e^{i[kL-(\pi/4)]}}{\sqrt{2\pi kL}} \frac{1}{k(1+\cos\theta)}\right] e^{ikL\cos\theta}. \quad (66)$$

Thus the total diffracted field with distances measured at origin $x=0$ and $z=0$, is given by

$$\phi^s(\rho, \theta) = [\phi_1(\rho, \theta) + \phi_{2,A}(\rho, \theta) + \phi_{2,B}(\rho, \theta)] e^{ika\sin\theta}. \quad (67)$$

B. Fields Inside the Exciting Waveguide

Again, the reflected field consists of the diffraction due to the exciting waveguide and the multiple diffraction between the two waveguides. However, the diffracted rays are now converted into modes inside the exciting waveguide. The reflection due to the open end of exciting waveguide is, Fig. 4(c),

$$\phi_{r,1}(x, z) = \sum_{m=1,3,5\cdots}^{\infty} [2i(-1)^{(m-1)/2}] \cdot \cos\left(\frac{m\pi}{2a} x\right) e^{\gamma_m z} \bar{D}[-(\pi - \phi_m), -(\pi - \phi_l)] [\text{ray to mode conversion factor}] E_y^i \quad (68)$$

where the first bracket is to normalize the amplitude of the rays traveling in the $-(\pi - \phi_m)$ direction at $x=d$, $z=0$, for an incident plane wave of unit amplitude at the above point. The second bracket is the ray to mode conversion factor given by [15]

$$\text{conversion factor} = \left(\left[\frac{d}{d\alpha}(\gamma\bar{G}(\alpha))\right]^{-1}\right)_{\alpha = k\cos\phi_m} = \frac{1}{2ka\cos\phi_m} \quad (69)$$

and E_y^i is given by (52). Equation (68), after some manipulations becomes

$$\phi_{r,1}(x, z) = \sum_{m=1,3,5,\cdots}^{\infty} R_{l,m} \cos\left(\frac{m\pi}{2a} x\right) e^{\gamma_m z} \quad (70)$$

with

$$R_{l,m} = \frac{-l\pi^2}{4a^3}(-1)^{(l+m)/2}\frac{mG_+(i\gamma_m)G_+(i\gamma_l)}{\gamma_m(\gamma_m+\gamma_l)}. \quad (71)$$

Similarly, the reflection due to rays of type A is shown as

$$\phi_{r,2,A}(x,z) = \sum_{m=1,3,5,\cdots}^{\infty} [2i(-1)^{(m-1)/2}] \cos\left(\frac{m\pi}{2a}x\right) \cdot e^{\gamma_m z}\left[\frac{\overline{F}}{1-\overline{F}^2}\overline{F}_i\overline{F}_{f_1}\right]$$

$$[\text{ray to mode conversion factor}] \quad (72)$$

where $\overline{F}$, $\overline{F}_i$, and $\overline{F}_{f_1}$ are given, respectively, by (61), (60), and (62) with θ being replaced by ϕ_m and $(e^{i(k\rho-\pi/4)})/(\sqrt{2\pi k\rho})$ being dropped in (63). Hence one finds

$$\phi_{r,2,A}(x,z) = \sum_{m=1,3,5,\cdots}^{\infty} R_{l,m}^{(A)} \cos\left(\frac{m\pi}{2a}x\right)e^{\gamma_m z} \quad (73)$$

with

$$R_{l,m}^{(A)} = \frac{-il\pi^2}{4a^3}(-1)^{(l+m)/2}\frac{mG_+(i\gamma_l)G_+(i\gamma_m)}{\gamma_m(i\gamma_l-k)G_+^2(k)} \cdot \frac{\overline{F}}{1-\overline{F}^2}\frac{e^{i[(kL-(\pi/4))]}}{\sqrt{2\pi kL}\,(k-i\gamma_m)}. \quad (74)$$

The field due to rays of type B is a radiated field and has the same form as (66) with $\cos\theta \simeq 1$. This field can be converted into a model series to give $R_{l,m}^{(B)}(z)$ similar to $R_{l,m}^{(2)}(z)$. This is due to the fact that the result obtained here is the same as that of (26) obtained by an application of the saddle-point method of integration in conjunction with the asymptotic form of $T(-\alpha)$.

C. Fields Inside the Coupled Waveguide

The transmitted fields in the coupled waveguide also consist of diffracted fields due to exciting waveguide and multiple diffraction between the two waveguides, Fig. 4(d). Here the diffraction due to exciting waveguide and rays of type A are scattering type and give transmission coefficients which are a function of z and can be treated similar to rays of type B in Section V-B. The remaining contribution comes from rays of type B which may be shown to be

$$\phi_{t,2,B}(x,z) = \sum_{m=1,3,5,\cdots}^{\infty} [2i(-1)^{(m-1)/2}] \cdot \cos\left(\frac{m\pi}{2a}x\right)e^{-\gamma_m(z-L)}\left[\frac{\overline{F}_i\overline{F}_{f_2}}{1-\overline{F}^2}\right]$$

$$[\text{ray to mode conversion factor}] \quad (75)$$

which could again be modified to

$$\phi_{t,2,B}(x,z) = \sum_{m=1,3,5,\cdots}^{\infty} T_{l,m}^{(B)}\left(\frac{m\pi}{2a}x\right)e^{-\gamma_m z} \quad (76)$$

with

$$T_{l,m}^{(B)} = \frac{-il\pi^2}{4a^3}(-1)^{(l+m)/2}\frac{mG_+(i\gamma_l)G_+(i\gamma_m)e^{\gamma_m L}}{\gamma_m(i\gamma_l-k)G_+^2(k)(1-\overline{F}^2)} \cdot \left[\frac{e^{i(kL-\pi/4)}}{\sqrt{2\pi kL}\,(k-i\gamma_m)}\right]. \quad (77)$$

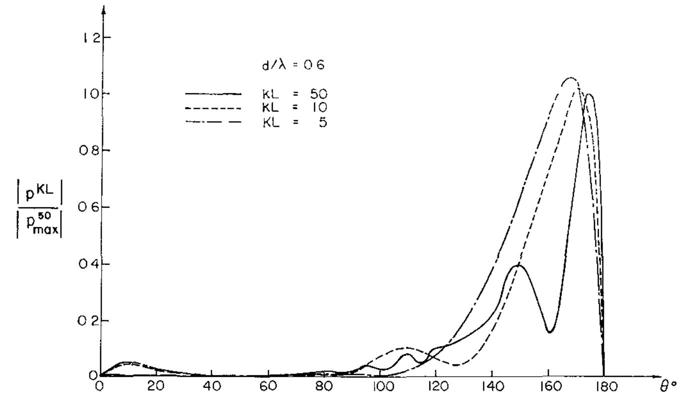

Fig. 5. Radiation pattern of the $TE_{0,1}$ mode.

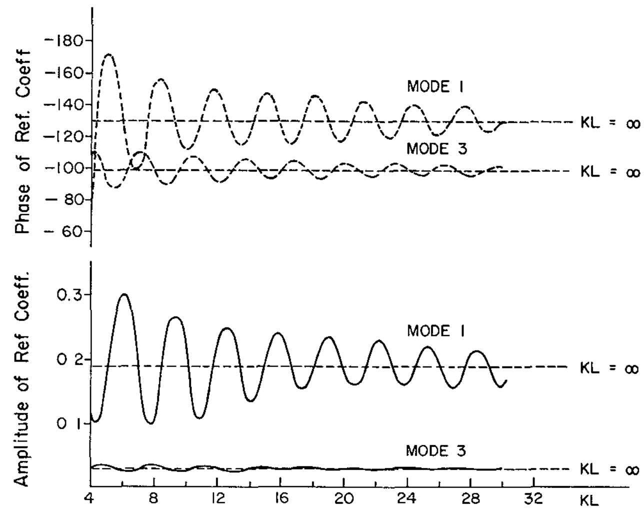

Fig. 6. Reflection coefficients of the $TE_{0,1}$ mode for $d/\lambda = 0.6$.

VI. Results and Discussions

Some results are obtained for a waveguide size $2a/\lambda = 0.6$ and $TE_{0,1}$ excitation. The resulting infinite integrals are computed by a Gauss–Laguerre quadrature formula with 15 intervals. Fig. 5 shows the radiation patterns for $kL = 5$, 10, and 50, which are normalized to the maximum power radiation at $kL = 50$. Since the radiation patterns are symmetric with respect to the waveguide geometry, only the patterns for $0 \leq \theta \leq 180°$ are presented. As expected, with decreasing kL, the direction of the radiation main lobe moves progressively away from the forward direction and the back-lobe level increases. The amount of radiated power, however, should be an oscillating function of kL similar to the reflection and transmission fields discussed below.

The reflection coefficients for modes 1 and 3 are shown in Fig. 6. Since $R_{l,m}^{(2)}(z)$ decays with z and does not contribute to the reflected field at large distances from the opening, its corresponding terms were not included in computations. The amplitude and phase of the reflection coefficients are oscillating functions of period π and decay continuously to reach the final values for $kL = \infty$, a single excited waveguide. Fig. 7 shows the transmission coefficient (coupling to coupled waveguide) for the first mode, which is again an oscillating function decaying to zero as kL approaches infinity. This transmission coefficient is again computed by neglecting the corresponding terms for scattered fields which vanish at large distances from the opening. The total reflection and transmission terms at the open ends of two waveguides are also computed and are shown

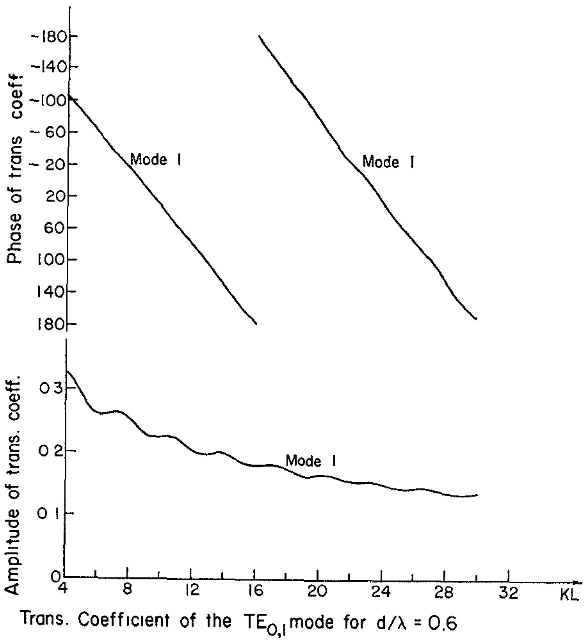

Fig. 7. Transmission coefficient of the $TE_{0,1}$ mode for $d/\lambda = 0.6$.

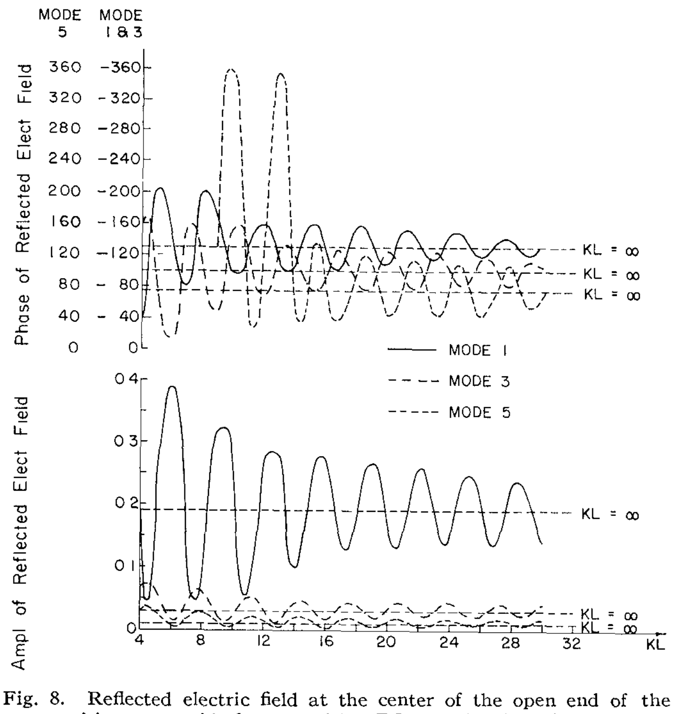

Fig. 8. Reflected electric field at the center of the open end of the exciting waveguide for an exciting $TE_{0,1}$ mode with $d/\lambda = 0.6$.

separately for modes 1, 3, and 5 in Figs. 8 and 9. These results show the relative magnitude of each mode at the open ends and may be used to find the resulting aperture fields.

Since similar analytic or experimental results were not located elsewhere in the literature, no comparison is possible at the present time. However, the behavior of the results are as expected. The analysis in this paper was carried out for a $TE_{0,l}$ excitation with l odd. The extension to $TE_{0,l}$ with l even and $TM_{0,l}$ with even or odd l is trivial and can be carried out with the proper Green's functions. The method can also

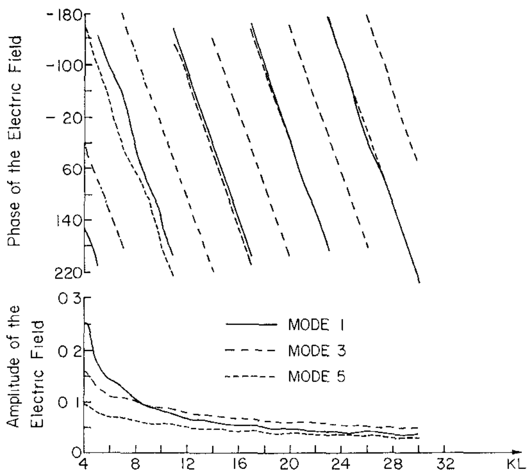

Fig. 9. Electric field at the center of the open end of the coupled waveguide for an exiciting $TE_{0,1}$ mode with $d/\lambda = 0.6$.

be extended to waveguides with different widths as well as coupling between waveguide arrays.

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